

Study materials of Mathematics for class D-III (H),
 Paper - VI on topic "Inner Product space"
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Distance in an inner product space:

Theorem: Let u, v be any two vectors of inner product space.
 We define $d(u, v)$, the distance from u to v as

$$d(u, v) = \|u - v\|$$

It is to prove That

(i) $d(u, v) \geq 0$ and $d(u, v) = 0$ iff $u = v$

(ii) $d(u, v) = d(v, u)$

(iii) $d(u, v) \leq d(u, w) + d(w, v)$

(iv) $d(u, v) = d(u+w, v+w)$

Proof of (i) $d(u, v) = \|u - v\| \geq 0$ and $= 0$ iff $u - v = 0 \Rightarrow u = v$

(ii) $d(u, v) = \|u - v\| = \|-(v - u)\| = |-1| \|v - u\|$

$$= \|v - u\| = d(v, u)$$

(iii) $d(u, v) = \|u - v\| = \|u - w + w - v\|$

$$\leq \|u - w\| + \|w - v\|$$

by Triangle inequality

$$= d(u, w) + d(w, v)$$

$$\therefore d(u, v) \leq d(u, w) + d(w, v)$$

(iv) $d(u, v) = \|u - v\| = \|(u+w) - (v+w)\|$

$$= d(u+w, v+w)$$

Hence d is a metric on the inner product space.

Example. If u and v be any two vectors in an inner product space, Show that $\|u+v\|^2 - \|u-v\|^2 = 4 \operatorname{Re}(u, v)$

Solution. We have

$$\|u+v\|^2 = (u+v, u+v)$$

$$= (u, u+v) + (v, u+v)$$

$$= (u, u) + (u, v) + (v, u) + (v, v)$$

$$= \|u\|^2 + (u, v) + (v, u) + \|v\|^2 \quad \text{--- ①}$$

$$\text{Also, } \|u-v\|^2 = (u-v, u-v)$$

$$= (u, u-v) - (v, u-v)$$

$$= (u, u) - (u, v) - (v, u) + (v, v)$$

$$= \|u\|^2 - (u, v) - (v, u) + \|v\|^2 \quad \text{--- ②}$$

Subtracting ② from ①, we get

$$\|u+v\|^2 + \|u-v\|^2 = 2(u, u) + 2(v, v)$$

$$= 2(u, u) + 2\overline{(u, v)}$$

$$= 2[(u, u) + \overline{(u, v)}]$$

$$= 2 \cdot 2 \operatorname{Re}(u, v)$$

$$= 4 \operatorname{Re}(u, v)$$

Parallelogram law:

if u and v are vectors in an inner product space,

Prove that $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

Proof:

We have $\|u+v\|^2 = (u+v, u+v)$

$$= (u, u+v) + (v, u+v)$$

$$= (u, u) + (u, v) + (v, u) + (v, v)$$

$$\text{①} \rightarrow \|u+v\|^2 = \|u\|^2 + (u, v) + (v, u) + \|v\|^2 \text{ --- ①}$$

Also, $\|u-v\|^2 = (u-v, u-v)$

$$= (u, u-v) - (v, u-v)$$

$$= (u, u) - (u, v) - (v, u) + (v, v)$$

$$= \|u\|^2 - (u, v) - (v, u) + \|v\|^2 \text{ --- ②}$$

Adding ① & ②, we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \text{ Proved.}$$

Polarization Identity

if u and v are vectors in an inner product ^{vector} space

Prove that

$$4(u, v) = \|u+iv\|^2 - \|u-iv\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2$$

Proof:

Subtracting ② from ①, we have

$$\|u+iv\|^2 - \|u-iv\|^2 = 2(u, v) + 2(v, u) \text{ --- ③}$$

Replacing v by iv in ③, we have

$$\|u+iv\|^2 - \|u-iv\|^2 = 2(u, iv) + 2(iv, u)$$

$$= 2i(u, v) + 2i(v, u)$$

$$= -2i(u, v) + 2i(v, u) \text{ --- ④}$$

Multiplying both sides of (4) by i , we get

$$\begin{aligned} i\|u+iv\|^2 - i\|u-iv\|^2 &= -2i^2(u,v) + 2i^2(v,u) \\ &= 2(u,v) - 2(v,u) \quad \text{--- (5)} \end{aligned}$$

Adding (3) & (5), we get

$$\|u+iv\|^2 - \|u-iv\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2 = 4(u,v) \quad \underline{\underline{\text{Proved.}}}$$

Orthogonality.

Orthogonal vectors:

Defⁿ: Two vectors $u, v \in V$ where V is an inner product space, are said to be orthogonal $\forall (u,v) = 0$

In other words, $u \perp v \Rightarrow (u,v) = 0$

Vector orthogonal to a subset S of V .

Let S be a subset of an inner product vector space V and let u be any vector in V .

Then u is said to be orthogonal to a set S i.e.

$u \perp S$ i.e. u is orthogonal to every vector in S .

i.e. $u \perp v \quad \forall v \in S$ i.e. $(u,v) = 0, \quad \forall v \in S$

Hence $u \perp S \Rightarrow (u,v) = 0, \quad \forall v \in S.$
