Study materials of Mathematics for class D-III (H), Paper-VI on topic "Inner Product space" of Abstract Algebra, composed by Dr. Md. Shamim Ahmad, Associate Professor, U. R. College Rosera (Samastipur)

Inner Product space

Defn: Let $V(F)$ be linear (vector) space where F is either the field of real numbers or the field of complex numbers. An inner product on V is a mapping from $V \times V$ into F which assigns to each ordered pair of vectors u, v in V a scalar (u,v) in F satisfying the following axioms:

$[IP_1]$ $(u,u) \geq 0$ and $(u,u) = 0$ if $u = 0$
(positive definite property)

$[IP_2]$ $(u,v) = \overline{(v,u)}$ (Conjugate symmetry property)

$[IP_3]$ $(au + bv, w) = a(u,w) + b(v,w)$
for any $u, v, w \in V$ and $a, b \in F$ (linear property)

Also, the linear space is then said to be an inner product space with respect to that specified inner product defined on it.

Theorem: Let $V(F)$ be an inner product space over F. Let $u, v, w \in V(F)$ & $\alpha, \beta \in F$, then

(i) $(u, v+w) = (u,v) + (u,w)$ (ii) $(u, \alpha v) = \bar{\alpha}(u,v)$

(iii) $(\alpha u, \beta v) = \alpha\bar{\beta}(u,v)$ (iv) $(0,v) = (u,0) = 0$

(2)

(v) $(u-v, w) = (u,w) - (v,w)$

(vi) $(u, v-w) = (u,v) - (u,w)$

(vii) $(au - bv, w) = a(u,w) - b(v,w)$

(viii) $(u, av+bw) = \bar{a}(u,v) + \bar{b}(u,w)$

(ix) $(u, av-bw) = \bar{a}(u,v) - \bar{b}(u,w)$

Proof of (i)

Using $[IP_2]$, we have

$$(u, v+w) = \overline{(v+w, u)} = \overline{(v,u) + (w,u)}$$
$$= \overline{(v,u)} + \overline{(w,u)}$$
$$= (u,v) + (u,w)$$

Proof of (ii) Using $[IP_2]$, we have

$$(u, \alpha v) = \overline{(\alpha v, u)} = \overline{\alpha (v,u)}$$
$$= \bar{\alpha}\,\overline{(v,u)} = \bar{\alpha}(u,v)$$

Proof of (iii)

from (ii) we have

$$(\alpha u, \beta v) = \bar{\beta}(\alpha u, v) = \bar{\beta}\alpha(u,v)$$
$$= \alpha\bar{\beta}(u,v)$$

Proof of (iv)

We have $0 + (0,v) = (0,v) = (0+0,v)$

$$= (0,v) + (0,v)$$

Therefore by right cancellation law in F, we have

$$(0,v) = 0$$

Similarly $(u,0) = 0$

(v) $(u-v,\ w) = (u+(-v),\ w)$

$= (u,w) + (-v, w)$

$= (u,w) + (-1)(v,w)$

$= (u,w) - (v,w)$

(vi) $(u,\ v-w) = \overline{(v-w, u)} = \overline{(v,u) - (w,u)}$

$= \overline{(v,u)} - \overline{(w,u)}$

$= (u,v) - (u,w)$

(vii) $(au-bv,\ w) = (au+(-b)v,\ w)$

$= a(u,w) + (-b)(v,w)$

$= u(u,w) - b(v,w)$

(viii) $(u,\ av+bw) = \overline{(av+bw, u)} = \overline{a(v,u) + b(w,u)}$

$= \overline{a(v,u)} + \overline{b(w,u)}$

$= \bar{a}\,\overline{(v,u)} + \bar{b}\,\overline{(w,u)}$

$= \bar{a}\,(u,v) + \bar{b}\,(u,w)$

(ix) $(u,\ av-bw) = (u,\ av+(-b)w)$

$= \bar{a}\,(u,v) + \overline{(-b)}\,(u,w)$

$= \bar{a}\,(u,v) - \bar{b}\,(u,w)$

Example 1. Let R^n be the real linear space of n-tuples of real numbers w.r.t. the usual Co-ordinatewise addition & scalar multiplication.

for any $x = (x_1, x_2, \ldots x_n) \in R^n$

$y = (y_1, y_2, \ldots, y_n) \in R^n$

let the inner product be defined as $(x, y) = \sum_{i=1}^{n} x_i y_i$.

Then R^n is an inner product space over R w.r.to the given definition of inner Product.

Solution. We shall verify all the Three conditions of an inner product space one by one.

By def^n, we have

(i) $(x, x) = \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} (x_i)^2 \geq 0$

Again $(x, x) = 0 \Leftrightarrow \sum_{i=1}^{n} x_i^2 = 0$

$\Leftrightarrow x_i^2 = 0$ for each $i = 1, 2, 3, \ldots, n$

$\Leftrightarrow x_i = 0$, for each $i = 1, 2, 3, \ldots, n$

$\Leftrightarrow x = 0$

Thus $[IP_1]$ is satisfied.

(ii) Again, $(x, y) = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = (y, x)$

$\therefore [IP_2]$ is satisfied.

(iii) Next if $a, b \in R$, Then by def^n, we have

$(ax + by, z) = \sum_{i=1}^{n} (ax_i + by_i) z_i$

$= \sum_{i=1}^{n} (ax_i z_i + by_i z_i) = a \sum_{i=1}^{n} x_i z_i + b \sum_{i=1}^{n} y_i z_i$

$= a(x, z) + b(y, z)$

Thus $[IP_3]$ is satisfied.

Hence the inner product satisfies the conditions from $[IP_1]$ to $[IP_3]$.

Therefore, R^n is an inner product space over R.