Mathematics
for
Class Deg. III (H)
Paper — VIII
(Spherical Trigonometry & Astronomy)
by

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Paper – VIII ① Spherical Trigonometry

Example In a spherical triangle ABC, show that

$$\sin b \cdot \sin c + \cos b \cdot \cos c \cdot \cos A = \sin B \cdot \sin C - \cos B \cdot \cos C \cdot \cos a$$

Sol^n. By Cosine formula, we have

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin C \cdot \cos A \quad \text{——— (1)}$$

& By Supplemental Cosine formula, we have

$$\cos A = -\cos B \cdot \cos C + \sin B \cdot \sin C \cdot \cos a \quad \text{——— (2)}$$

$$\begin{aligned}
\text{L.H.S} &= \sin b \cdot \sin c + \cos b \cdot \cos c \cdot \cos A \\
&= \sin b \cdot \sin c + (\cos a - \sin b \cdot \sin c \cdot \cos A) \cdot \cos A \quad [\text{from (1)}] \\
&= \sin b \cdot \sin c - \sin b \cdot \sin c \cdot \cos^2 A + \cos a \cdot \cos A \\
&= \sin b \cdot \sin c\,(1 - \cos^2 A) + \cos a \cdot \cos A \\
&= \sin b \cdot \sin c \cdot \sin^2 A + \cos a \cdot \cos A \\
&= \sin b \cdot \sin c \cdot \sin^2 A + \cos a\,(-\cos B \cdot \cos C + \sin B \cdot \sin C \cdot \cos a) \quad \text{from (2)} \\
&= \sin b \cdot \sin c \cdot \sin^2 A - \cos a \cdot \cos B \cdot \cos C + \sin B \cdot \sin C \cdot \cos^2 a \quad \text{——— (3)}
\end{aligned}$$

By sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\therefore \sin b \cdot \sin A = \sin a \cdot \sin B$$

$$\& \quad \sin c \cdot \sin A = \sin a \cdot \sin C$$

Multiplying these we have

$$\sin b \cdot \sin c \cdot \sin^2 A = \sin^2 a \cdot \sin B \cdot \sin C$$

Hence (3) becomes

$$\begin{aligned}
\text{L.H.S} &= \sin^2 a \cdot \sin B \cdot \sin C - \cos a \cdot \cos B \cdot \cos C + \sin B \cdot \sin C \cdot \cos^2 a \\
&= (\sin^2 a + \cos^2 a) \sin B \cdot \sin C - \cos B \cdot \cos C \cdot \cos a \\
&= 1 \cdot \sin B \cdot \sin C - \cos B \cdot \cos C \cdot \cos a = \sin B \cdot \sin C - \cos B \cdot \cos C \cdot \cos a \\
&= \text{R.H.S} \quad \text{Proved.}
\end{aligned}$$

Example If a spherical triangle be equal & similar to its polar triangle show that $\sec^2 A + \sec^2 B + \sec^2 C + 2\sec A . \sec B . \sec C = 1$ and deduce that such a triangle cannot be equilateral.

Solution. Let ABC be the spherical triangle and A′B′C′ its polar triangle, then since $\Delta ABC \equiv \Delta A'B'C'$

We have $A = A' = \pi - a$ & $a = a' = \pi - A$

$B = B' = \pi - b$ & $b = b' = \pi - B$

$C = C' = \pi - c$ & $c = c' = \pi - C$

By supplemental cosine formula in ΔABC, we have

$$\cos A = -\cos B . \cos C + \sin B . \sin C . \cos a$$

$$\Rightarrow \cos A = -\cos B . \cos C + \sin B . \sin C . \cos(\pi - A)$$

$$= -\cos B . \cos C - \sin B . \sin C . \cos A$$

Dividing by $\cos A . \cos B . \cos C$, we get

$$\frac{1}{\cos B . \cos C} = -\frac{1}{\cos A} - \frac{\sin B}{\cos B} . \frac{\sin C}{\cos C}$$

$$\Rightarrow \sec B . \sec C = -\sec A - \tan B . \tan C$$

$$\Rightarrow \sec A + \sec B . \sec C = -\tan B . \tan C$$

Squaring both sides, we have

$$(\sec A + \sec B . \sec C)^2 = \tan^2 B \tan^2 C$$

$$\Rightarrow \sec^2 A + \cancel{\sec^2 B \sec^2 C} + 2\sec A . \sec B \sec C = (\sec^2 B - 1)(\sec^2 C - 1)$$

$$= \cancel{\sec^2 B . \sec^2 C} - \sec^2 B - \sec^2 C + 1$$

$$\Rightarrow \sec^2 A + \sec^2 B + \sec^2 C + 2\sec A . \sec B . \sec C = 1 \quad \text{——} (1)$$

Proved first result.

Now, if the triangle be equilateral, then

$A = B = C$

Then (1) becomes

$$3\sec^2 A + 2\sec^3 A = 1$$

$$\Rightarrow 2\sec^3 A - \sec^2 A = 1 - 4\sec^2 A$$

$$\Rightarrow \sec^2 A(2\sec A - 1) = (1 - 2\sec A)(1 + 2\sec A)$$

$$\Rightarrow (2\sec A - 1)\left[\sec^2 A + 1 + 2\sec A\right] = 0$$

$$\Rightarrow (2\sec A - 1)(\sec A + 1)^2 = 0$$

$\therefore$ Either $2\sec A = 1$ i.e. $\cos A = 2$

OR $\sec A + 1 = 0$ i.e. $\cos A = -1$

As $\cos A$ cannot be 2, Therefore

$$\cos A = -1 = \cos\pi$$

$$\therefore A = \pi = B = C$$

$\therefore A + B + C = 3\pi$ i.e. six right angles which is also not possible as we know that in a spherical Triangle, the sum of the angles of a spherical Triangle is less than six right angles.

Hence such a triangle cannot be equilateral.