Subject	Class	Paper	Study materials on	Resource Person
Mathematics	D-I (H)	I	Cardon's method of solving Cubic Eqn	Dr. S. Ahmed

Explain Cardan's method of solving Cubic Equation

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Let $a_0x^3+3a_1x^2+3a_2x+a_3=0$ ——— (1)

be the cubic equation.

The above equation can be reduced to the form

$$z^3+3Hz+G=0 \quad \text{——— (2)}$$

Where $z=a_0x+a_1$, $H=a_0a_2-a_1^2$, $G=a_0^2a_3-3a_0a_1a_2+2a_1^3$

Now, let us assume $z=u+v$ be the Soln of (2)

$$\therefore z^3=(u+v)^3=u^3+v^3+3uv(u+v)$$

$$\Rightarrow z^3-3uvz-(u^3+v^3)=0 \quad \text{——— (3)}$$

Comparing (2) & (3), we have

$$uv=-H \quad \text{——— (4)}$$

and $u^3+v^3=-G$

$$\therefore u^3v^3=-H^3$$

Hence u^3 & v^3 are the roots of the quadratic

$$t^2+Gt-H^3=0$$

$$\therefore t=\frac{-G\pm\sqrt{G^2+4H^3}}{2}$$

Hence $u^3=\dfrac{-G+\sqrt{G^2+4H^3}}{2}$ & $v^3=\dfrac{-G-\sqrt{G^2+4H^3}}{2}$

Now, from above we shall obtain three values of u and three of v, our root being $u+v$ will therefore nine possible values because each of three values of u is to be added to any one of three values of v. But there is limitation imposed by the relation (4)

i.e. $uv=-H=$ Constant or $v=\dfrac{-H}{u}$

Hence the values of u and v to be combined should be such as to satisfy $uv=-H$. Now, if we extract cube root of u^3, we get $u, u\omega, u\omega^2$ and that of v^3 gives $v, v\omega, v\omega^2$.

Hence the three roots of the equation are

$u+v,\ u\omega+v\omega^2,\ u\omega^2+v\omega$

Where $\omega = \frac{-1+\sqrt{-3}}{2}$ & $\omega^2 = \frac{-1-\sqrt{-3}}{2}$

Having found z, we can find the values of x by the relation

$Z = a_0 x + a_1$

Example: Solve $x^3-15x^2-33x+847=0$

The given equation can be written as

$$x^3+3\cdot(-5)x^2+3\cdot(-11)x+847=0 \quad \text{— (1)}$$

It is of the form $a_0x^3+3a_1x^2+3a_2x+a_3=0$

Where $a_0=1,\ a_1=(-5),\ a_2=-11,\ a_3=847$

Now, $H = a_0a_2-a_1^2$

$= 1\cdot(-11)-(-5)^2 = -11-25 = -36$

$G = a_0^2a_3-3a_0a_1a_2+2a_1^3$

$= (1)^2\cdot 847-3\cdot1\cdot(-5)(-11)+2\cdot(-5)^3$

$= 847-165-250 = 847-315 = 432.$

Hence the transformed equation is

$$Z^3+3HZ+G=0$$

$$\Rightarrow z^3+3(-36)z+432=0$$

$$\Rightarrow z^3-108z+432=0 \quad \text{—— (2)}$$

Let $z = u+v$

$\therefore z^3 = (u+v)^3 = u^3+v^3+3uv(u+v)$

$\Rightarrow z^3-3uvz-(u^3+v^3)=0$ —— (3)

Comparing (2) & (3), we have

$uv = 36$ & $u^3+v^3 = -432$

$\therefore u^3$ & v^3 are the roots of $t^2+432t+(36)^3=0$

$\Rightarrow t^2+2\cdot216\cdot t+(6^3)^2=0$

$\Rightarrow (t+6^3)^2=0$

$\therefore t = -6^3, -6^3$

$\Rightarrow u^3 = -6^3 \Rightarrow u = -6, v = -6$

$\therefore u+v = -12$ is a root of the z-cubic when divided by $z+12$ gives the quadratic

$$z^2 - 12z + 36 = 0$$

$$\Rightarrow (z-6)^2 = 0$$

$\therefore$ roots of z-cubic are 6, 6, −12 and hence of x-cubic are 11, 11, −7 (two roots equal).

Q. Solve $28x^3 - 9x^2 + 1 = 0$

Soln. Putting $x = \frac{1}{z}$, we get

$$\frac{28}{z^3} - \frac{9}{z^2} + 1 = 0$$

$$\Rightarrow z^3 - 9z + 28 = 0$$

Let $z = u+v$

$\therefore z^3 = u^3 + v^3 + 3uv(u+v)$

$\Rightarrow z^3 - 3uvz - (u^3+v^3) = 0$, Comparing, we get $uv = 3 \Rightarrow u^3v^3 = 27$

$u^3 + v^3 = -28$

$\therefore u^3$ & v^3 are the roots of

$$t^2 + 28t + 27 = 0$$

$$\Rightarrow t^2 + 27t + t + 27 = 0$$

$$\Rightarrow t(t+27) + 1(t+27) = 0 \Rightarrow (t+1)(t+27) = 0$$

$$\therefore t = -1, -27$$

$$\therefore u^3 = -27, v^3 = -1$$

$$z = u+v = -3-1 = -4$$

When $z^3 - 9z + 28$ is divided by $z+4$ giving us $z^2 - 4z + 7$

$\cancel{z^3 - 9z + 28}$ $\therefore z^3 - 9z + 28 = (z+4)(z^2 - 4z + 7)$

$\therefore z = -4, \frac{4 \pm \sqrt{16-28}}{2}$ ie $2 \pm i\sqrt{3}$

$$\therefore\ x = \frac{1}{z} = -\frac{1}{4},\ \frac{1}{2+i\sqrt{3}},\ \frac{1}{2-i\sqrt{3}}$$

$$= -\frac{1}{4},\ \frac{2-i\sqrt{3}}{7},\ \frac{2+i\sqrt{3}}{7}$$