Mathematics
for
Class Deg. III (H)
Paper — VIII
(Spherical Trigonometry & Astronomy)

by

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Example. In any equilateral triangle ABC, show that

(i) $2\cos\frac{a}{2}\cdot\sin\frac{A}{2}=1$

(ii) $1+2\cos a = \cot^2 A/2$

(iii) $\tan^2\frac{a}{2} = 1-2\cos A$

Solution

In an equilateral spherical triangle $a=b=c$ & $A=B=C$

Now, by cosine formula we have

$$\cos A = \frac{\cos a - \cos b\cdot\cos c}{\sin b\cdot\sin c}$$

$$= \frac{\cos a - \cos a\cdot\cos a}{\sin a\cdot\sin a} \quad [\because a=b=c]$$

$$= \frac{\cos a-\cos^2 a}{\sin^2 a}$$

$$= \frac{\cos a(1-\cos a)}{1-\cos^2 a} = \frac{\cos a\,\cancel{(1-\cos a)}}{\cancel{(1-\cos a)}(1+\cos a)} = \frac{\cos a}{1+\cos a}$$

$$\therefore \cos A = \frac{\cos a}{1+\cos a} \quad \text{———— (1)}$$

$$\Rightarrow 1-2\sin^2\frac{A}{2} = \frac{\cos a}{1+\cos a}$$

$$\Rightarrow 2\sin^2\frac{A}{2} = 1-\frac{\cos a}{1+\cos a} = \frac{1+\cancel{\cos a}-\cancel{\cos a}}{1+\cos a} = \frac{1}{1+\cos a}$$

$$= \frac{1}{2\cos^2\frac{a}{2}}$$

$$\Rightarrow 4\cos^2\frac{a}{2}\cdot\sin^2\frac{A}{2}=1$$

$$\therefore 2\cos\frac{a}{2}\cdot\sin\frac{A}{2}=1 \quad \text{Proved (i)}$$

From ①, we have $\cos A = \frac{\cos a}{1+\cos a}$.

$$\Rightarrow 1-\cos A = 1-\frac{\cos a}{1+\cos a} = \frac{1}{1+\cos a} \quad \therefore \frac{1}{1-\cos A} = 1+\cos a$$

$$\therefore \cos a = \frac{\cos A}{1-\cos A}$$

(2)

$$1+2\cos a = 1+\frac{2\cos A}{1-\cos A} = \frac{1+\cos A}{1-\cos A} = \frac{2\cos^2 A/2}{2\sin^2 A/2} = \cot^2\frac{A}{2}.$$

Hence $\boxed{1+2\cos a = \cot^2\frac{A}{2}}$ Proved (ii)

Again from (1), we have

$$\cos A = \frac{\cos a}{1+\cos a}.$$

$$1-2\cos A = 1-\frac{2\cos a}{1+\cos a} = \frac{1-\cos a}{1+\cos a} = \frac{2\sin^2\frac{a}{2}}{2\cos^2\frac{a}{2}} = \tan^2\frac{a}{2}$$

$\therefore \boxed{\tan^2\frac{a}{2} = 1-2\cos A}$ Proved (iii)

mple In a spherical triangle ABC, $a=b=\frac{\pi}{3}$ & $c=\frac{\pi}{2}$

Prove that $A+B+C = \pi + \cos^{-1}(7/9)$

By cosine formula, we have

$$\cos A = \frac{\cos a - \cos b\cdot\cos c}{\sin b\cdot\sin c}$$

$$\& \cos C = \frac{\cos c - \cos a\cdot\cos b}{\sin a\cdot\sin b}$$

Now, given that $a=b=\pi/3$ & $c=\pi/2$

$$\therefore \cos A = \frac{\cos\frac{\pi}{3} - \cos\frac{\pi}{3}\cdot\cos\pi/2}{\sin\frac{\pi}{3}\cdot\sin\frac{\pi}{2}} = \frac{\frac{1}{2}-\frac{1}{2}\cdot 0}{\frac{\sqrt{3}}{2}\cdot 1} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\therefore \cos A = \frac{1}{\sqrt{3}} = \cos B \quad \left[\begin{array}{l}\text{as } a=b\\ \therefore A=B\end{array}\right]$$

Also, $$\cos C = \frac{\cos c - \cos a\cdot\cos b}{\sin a\cdot\sin b} = \frac{\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}} = \frac{0-\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}$$

$$= -\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 = -\frac{1}{3}$$

$$\therefore \cos C = -\frac{1}{3}$$

(3)

Now, $\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{1-\frac{1}{3}} = \sqrt{\frac{2}{3}}$

$\therefore \sin A = \sqrt{\frac{2}{3}} = \sin B$

and $\sin C = \sqrt{1-\cos^2 C} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

Hence $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$= \sqrt{\frac{2}{3}} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \sqrt{\frac{2}{3}}$

$= \frac{2\sqrt{2}}{3}$

& $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} - \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{2}{3}}$

$= \frac{1}{3} - \frac{2}{3} = \frac{-1}{3}$

$\therefore \cos\{(A+B)+C\} = \cos(A+B) \cdot \cos C - \sin(A+B) \cdot \sin C$

$= -\frac{1}{3} \cdot \left(-\frac{1}{3}\right) - \frac{2\sqrt{2}}{3} \cdot \frac{2\sqrt{2}}{3}$

$= \frac{1}{9} - \frac{8}{9} = \frac{-7}{9} = \cos\{\pi + \cos^{-1}(7/9)\}$

$\therefore A+B+C = \pi + \cos^{-1}(7/9)$

Proved.