Subject	Class	Paper	Study materials on	Resource Person	
Mathematics	D-III (H)	V	Complex Analysis IV	Dr. S. Ahmad	26.4.20

Limit of a function of a Complex variable

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0, then $f(z)$ is said to have a limit l as z approaches z_0 along any path if given an arbitrary real number $\epsilon > 0$, however small there exists a real number $\delta > 0$ such that

$$|f(z) - l| < \epsilon \quad \text{Whenever} \quad 0 < |z - z_0| < \delta$$

In symbolic form, $\lim_{z \to z_0} f(z) = l$

Note (i) δ usually depends upon ϵ

(ii) $z \to z_0$ implies that z approaches z_0 along any path.

The limits must be independent of the manner in which z approaches z_0.

If we get two different limits as $z \to z_0$ along two different paths, then limit does not exist.

Continuity

The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for given positive number ϵ, we can find a number δ such that $|f(z) - f(z_0)| < \epsilon$ for all points z of the domain satisfying

$$|z - z_0| < \delta$$

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Differentiability

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided that the limit exists and is dependent of the path along which $\delta z \to 0$.

Theorem Continuity is a necessary condition but not sufficient condition for the existence of a finite derivative.

or

$$\text{differentiability} \Rightarrow \text{Continuity}$$

Proof: We have $f(z_0+\delta z) - f(z_0) = \delta z . \left[\frac{f(z_0+\delta z) - f(z_0)}{\delta z}\right]$ — (1)

Taking limit of both sides of (1), as $\delta z \to 0$, we have

$$\lim_{\delta z \to 0}\left[f(z_0+\delta z) - f(z_0)\right] = \lim_{\delta z \to 0} \delta z . \lim_{\delta z \to 0}\left[\frac{f(z_0+\delta z) - f(z_0)}{\delta z}\right]$$

$$= 0 . f'(z) = 0$$

$$\Rightarrow \lim_{\delta z \to 0}\left[f(z_0+\delta z) - f(z_0)\right] = 0$$

$$\Rightarrow \lim_{z \to z_0}\left[f(z) - f(z_0)\right] = 0 \Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$$

$\Rightarrow f(z)$ is Continuous at $z = z_0$.

The Converse of the above theorem is not true.

This can be shown by the following example.

Example Prove that the function $f(z) = |z|^2$ is Continuous everywhere but nowhere differentiable except at the origin.

Soln. Here, $f(z) = |z|^2 = x^2+y^2$ where $z = x+iy \therefore |z| = \sqrt{x^2+y^2}$

since x^2 & y^2 are polynomial, so x^2+y^2 is Continuous everywhere, therefore, $|z|^2$ is Continuous everywhere.

(3)

Now, we have $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$

$$= \lim_{\delta z \to 0} \frac{|z+\delta z|^2 - |z|^2}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z+\delta z)\overline{(z+\delta z)} - z\bar{z}}{\delta z} \quad [\because |z|^2 = z\bar{z}]$$

$$= \lim_{\delta z \to 0} \frac{(z+\delta z)(\bar{z}+\overline{\delta z}) - z\bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\cancel{z\bar{z}} + z\overline{\delta z} + \delta z \cdot \bar{z} + \delta z \cdot \overline{\delta z} - \cancel{z\bar{z}}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{z \cdot \overline{\delta z} + \delta z \cdot \bar{z} + \delta z \cdot \overline{\delta z}}{\delta z}$$

$$= \lim_{\delta z \to 0} \left[z \cdot \frac{\overline{\delta z}}{\delta z} + \bar{z} + \overline{\delta z} \right]$$

$$= \lim_{\delta z \to 0} \left[z \cdot \frac{\overline{\delta z}}{\delta z} + \bar{z} \right] \quad [\because \delta z \to 0 \Rightarrow \overline{\delta z} \to 0] \longrightarrow (1)$$

Let $\delta z = r(\cos\theta + i\sin\theta)$

$\overline{\delta z} = r(\cos\theta - i\sin\theta)$

$$\therefore \frac{\overline{\delta z}}{\delta z} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} = (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)^{-1}$$

$$= (\cos\theta - i\sin\theta)(\cos\theta - i\sin\theta)$$

$$= (\cos\theta - i\sin\theta)^2 = \cos 2\theta - i\sin 2\theta$$

$$\therefore \frac{\overline{\delta z}}{\delta z} = \cos 2\theta - i\sin 2\theta$$

Since $\frac{\overline{\delta z}}{\delta z}$ depends upon θ. It means for different values of θ, $\frac{\overline{\delta z}}{\delta z}$ has different values.

It means $\frac{\overline{\delta z}}{\delta z}$ has different values for different z ✱

✱ Therefore $\lim_{\delta z \to 0} \frac{\overline{\delta z}}{\delta z}$ does not tend to a unique limit when $z \neq 0$

Therefore from (1), it follows that $f'(z)$ is not unique and hence $f(z)$ is not differentiable when $z \neq 0$

But when $z = 0$, then $f'(z) = 0$ i.e. $f'(0) = 0$ and is unique;

Hence the function is differentiable at $z = 0$