Ex. 3 In any spherical triangle ABC, Prove that

(i) $$\frac{\sin(A+B)}{\sin C} = \frac{\cos a + \cos b}{1+\cos c}$$

(ii) $$\frac{\sin(a+b)}{\sin c} = \frac{\cos A + \cos B}{1 - \cos C}$$

Sol (i) By Cosine formula, we have

$$\cos A = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

$$\Rightarrow \cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \, \cos A \quad \text{—— (1)}$$

$$\& \quad \cos B = \frac{\cos b - \cos a \cdot \cos c}{\sin a \cdot \sin c}$$

$$\Rightarrow \cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B \quad \text{—— (2)}$$

Adding (1) & (2), we get

$$\cos a + \cos b = \cos c(\cos a + \cos b) + \sin c \,(\sin b \cdot \cos A + \sin a \cdot \cos B)$$

$$\Rightarrow (\cos a + \cos b)(1 - \cos c) = \sin c \,(\sin b \cdot \cos A + \sin a \cdot \cos B) \quad \text{—— (3)}$$

By Sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\therefore \sin b = \frac{\sin B}{\sin C} \cdot \sin c \quad \& \quad \sin a = \frac{\sin A}{\sin C} \cdot \sin c$$

Putting these values of $\sin a$ & $\sin b$ in (3), we get

$$(\cos a + \cos b)(1 - \cos c) = \sin c \cdot \left[\frac{\sin B}{\sin C} \cdot \sin c \cdot \cos A + \frac{\sin A}{\sin C} \cdot \sin c \cdot \cos B\right]$$

$$= \frac{\sin^2 c}{\sin C} \left[\sin A \cdot \cos B + \cos A \cdot \sin B\right]$$

$$= \frac{\sin^2 c}{\sin C} \cdot \sin(A+B)$$

②

$$\therefore \frac{\sin(A+B)}{\sin C} = \frac{(\cos a+\cos b)(1-\cos c)}{\sin^2 c}$$

$$= \frac{(\cos a+\cos b)(1-\cos c)}{1-\cos^2 c} = \frac{(\cos a+\cos b)\cancel{(1-\cos c)}}{\cancel{(1-\cos c)}(1+\cos c)}$$

$$= \frac{\cos a+\cos b}{1+\cos c}$$

Hence $\frac{\sin(A+B)}{\sin C} = \frac{\cos a+\cos b}{1+\cos c}$ Proved

(ii) By Supplemental cosine formula, we have

$$\cos A = -\cos B.\cos C + \sin B.\sin C.\cos a$$

$$\cos B = -\cos A.\cos C + \sin A.\sin C.\cos b$$

$$\therefore \cos A+\cos B = -(\cos A+\cos B).\cos C + \sin C[\sin B.\cos a + \sin A.\cos b]$$

$$\Rightarrow (\cos A+\cos B)(1+\cos C) = \sin C.(\sin B.\cos a + \sin A.\cos b) \quad \text{——} ①$$

Now, by sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\therefore \sin B = \frac{\sin b}{\sin c}.\sin C \quad \& \quad \sin A = \frac{\sin a}{\sin c}.\sin C$$

Using these in ①, we have

$$(\cos A+\cos B)(1+\cos C) = \sin C.\left[\frac{\sin b}{\sin c}.\sin C.\cos a + \frac{\sin a}{\sin c}.\sin C.\cos b\right]$$

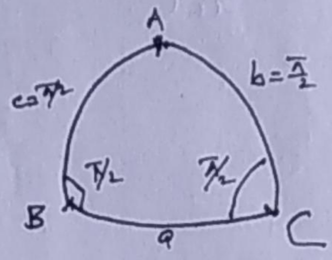
$$= \frac{\sin^2 C}{\sin c}\left[\sin a.\cos b + \cos a.\sin b\right] = \frac{\sin(a+b)}{\sin c}$$

$$\Rightarrow \frac{(\cos A+\cos B)(1+\cos C)}{\sin^2 C} = \frac{1}{\sin c}.\sin(a+b)$$

$$\Rightarrow \frac{\cos A+\cos B}{1-\cos C} = \frac{\sin(a+b)}{\sin c}$$

Proved

Example. If $b+c=\pi$, show that $\sin 2B + \sin 2C = 0$



By sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = K \text{ (say)}$$

$$\left.\begin{aligned} \therefore \sin B &= K \sin b \\ \sin C &= K \sin c \end{aligned}\right] \text{ — (1)}$$

Also, by Cosine formula, we have

$$\left.\begin{aligned} \cos B &= \frac{\cos b - \cos a \cdot \cos c}{\sin a \cdot \sin c} \\ \cos C &= \frac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \sin b}, \end{aligned}\right] \text{ — (2)}$$

Now, $\sin 2B + \sin 2C$

$$= 2\sin B \cdot \cos B + 2\sin C \cdot \cos C = 2[\sin B \cdot \cos B + \sin C \cdot \cos C]$$

$$= 2\left[K \sin b \cdot \frac{\cos b - \cos a \cdot \cos c}{\sin a \cdot \sin c} + K \sin c \cdot \frac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \sin b}\right]$$

$$= 2K\left[\cancel{\sin c} \cdot \frac{\cos b - \cos a \cdot \cos c}{\sin a \cdot \cancel{\sin c}} + \cancel{\sin b}\, \frac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \cancel{\sin b}}\right]$$

$$\left[\begin{aligned} \because b+c &= \pi \\ b &= \pi - c \\ \sin b &= \sin(\pi - c) = \sin c \end{aligned}\right]$$

$$= \frac{2K}{\sin a}[\cos b - \cos a \cdot \cos c + \cos c - \cos a \cdot \cos b]$$

$$\begin{aligned} \because b+c &= \pi \\ b &= \pi - c \\ \cos b &= \cos(\pi - c) = -\cos c \end{aligned}$$

$$= \frac{2K}{\sin a}[-\cos c - \cos a \cdot \cos c + \cos c - \cos a \cdot (-\cos c)]$$

$$= \frac{2K}{\sin a}[-\cancel{\cos c} - \cancel{\cos a \cdot \cos c} + \cancel{\cos c} + \cancel{\cos a \cdot \cos c}]$$

$$= \frac{2K}{\sin a} \times 0 = 0$$ **Proved.**