Subject	Class	Paper	Study materials on	Resource Person	Date
Mathematics	D-III (H)	V	Complex Analysis III	Dr. S. Ahmad	

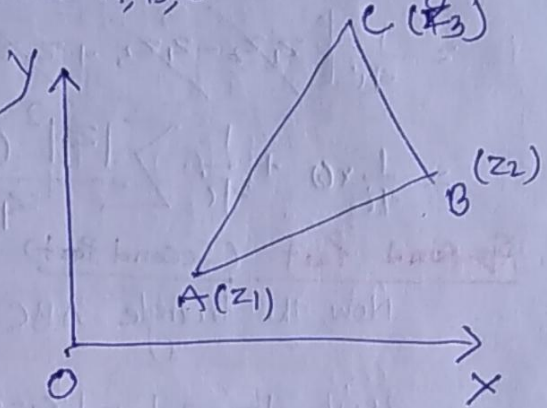
Example Prove that the area of the triangle whose vertices are the points z_1, z_2, z_3 on the Argand diagram is $\sum \frac{|z_1|^2 (z_2 - z_3)}{4 i z_1}$.

Show also that the triangle is equilateral if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Sol^n

Let z_1, z_2, z_3 represent the points A, B, C on the Argand diagram.



Also, let $z_1 = x_1 + i y_1 \equiv (x_1, y_1)$

$z_2 = x_2 + i y_2 \equiv (x_2, y_2)$

$z_3 = x_3 + i y_3 \equiv (x_3, y_3)$

Then the area of triangle Δ is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2i} \begin{vmatrix} x_1 & i y_1 & 1 \\ x_2 & i y_2 & 1 \\ x_3 & i y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i} \begin{vmatrix} x_1 & x_1 + i y_1 & 1 \\ x_2 & x_2 + i y_2 & 1 \\ x_3 & x_3 + i y_3 & 1 \end{vmatrix} \quad \text{operating } C_2 + C_1$$

$$= \frac{1}{2i} \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2i} \left[x_1 (z_2 - z_3) - x_2 (z_1 - z_3) + x_3 (z_1 - z_2) \right]$$

$$= \frac{1}{2i} \left[x_1 (z_2 - z_3) + x_2 (z_3 - z_1) + x_3 (z_1 - z_2) \right]$$

$$= \frac{1}{2i} \sum x_1 (z_2 - z_3) = \frac{1}{2i} \sum \frac{(z_1 + \bar{z}_1)}{2} . (z_2 - z_3)$$

$$\Delta = \frac{1}{4i}\sum(z_1+\bar{z}_1)(z_2-z_3)$$

$$= \frac{1}{4i}\sum\left[z_1(z_2-z_3)+\bar{z}_1(z_2-z_3)\right]$$

$$= \frac{1}{4i}\sum z_1(z_2-z_3) + \frac{1}{4i}\sum \bar{z}_1(z_2-z_3)$$

$$= \frac{1}{4i}\left[z_1(z_2-z_3)+z_2(z_3-z_1)+z_3(z_1-z_2)\right] + \frac{1}{4i}\sum\frac{z_1\bar{z}_1}{z_1}(z_2-z_3)$$

$$= \frac{1}{4i}\left[\cancel{z_1z_2}-\cancel{z_1z_3}+\cancel{z_2z_3}-\cancel{z_1z_2}+\cancel{z_1z_3}-\cancel{z_2z_3}\right] + \frac{1}{4i}\sum\frac{|z_1|^2}{z_1}(z_2-z_3)$$

$$= \frac{1}{4i}\times 0 + \frac{1}{4i}\sum\frac{|z_1|^2(z_2-z_3)}{z_1} = \sum\frac{|z_1|^2(z_2-z_3)}{4iz_1}$$ Proved.

Second Part (Second Part)

Now, the triangle ABC will be equilateral if $AB = BC = CA$

i.e. if $|z_1-z_2| = |z_2-z_3| = |z_3-z_1|$

$$\Rightarrow |z_1-z_2|^2 = |z_2-z_3|^2 = |z_3-z_1|^2$$

$$\Rightarrow (z_1-z_2)\overline{(z_1-z_2)} = (z_2-z_3)\overline{(z_2-z_3)} = (z_3-z_1)\overline{(z_3-z_1)}$$

$$\Rightarrow (z_1-z_2)(\bar{z}_1-\bar{z}_2) = (z_2-z_3)(\bar{z}_2-\bar{z}_3) = (z_3-z_1)(\bar{z}_3-\bar{z}_1) \quad \text{--- (1)}$$

From 1st two of (1), we have

$$\frac{z_1-z_2}{\bar{z}_2-\bar{z}_3} = \frac{z_2-z_3}{\bar{z}_1-\bar{z}_2} = \frac{(z_1-\cancel{z_2})+(\cancel{z_2}-z_3)}{(\cancel{\bar{z}_2}-\bar{z}_3)+(\bar{z}_1-\cancel{\bar{z}_2})}$$

$$\Rightarrow \frac{z_1-z_2}{\bar{z}_2-\bar{z}_3} = \frac{z_1-z_3}{\bar{z}_1-\bar{z}_3} \quad \text{--- (2)}$$

Again from last two of (1), we have

$$(z_2-z_3)(\bar{z}_2-\bar{z}_3) = (z_3-z_1)(\bar{z}_3-\bar{z}_1) \quad \text{--- (3)}$$

Multiplying (2) & (3), we have

$$\Rightarrow \frac{z_1-z_2}{\cancel{\bar z_2-\bar z_3}} \times (z_2-z_3)\cancel{(\bar z_2-\bar z_3)} = \frac{z_1-z_3}{\bar z_1-\bar z_3} \times (z_3-z_1)(\bar z_3-\bar z_1)$$

$$\Rightarrow (z_1-z_2)(z_2-z_3) = \frac{(z_1-z_3)}{\cancel{\bar z_1-\bar z_3}} \times -(z_1-z_3)\cdot -\cancel{(\bar z_1-\bar z_3)}$$

$$\Rightarrow (z_1-z_2)(z_2-z_3) = (z_1-z_3)^2$$

$$\Rightarrow z_1z_2 - z_1z_3 - z_2^2 + z_2z_3 = z_1^2 + z_3^2 - 2z_1z_3$$

$$\Rightarrow z_1^2 + z_3^2 + z_2^2 = z_1z_2 - z_1z_3 + z_2z_3 + 2z_1z_3$$

$$\Rightarrow \boxed{z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1}$$

It is required Condition.

Example. Find the loci of the point z satisfying the condition $\left|\frac{z-i}{z+i}\right| \geq 2$

Here, we have $\left|\frac{z-i}{z+i}\right| \geq 2$

$$\Rightarrow \left|\frac{z-i}{z+i}\right|^2 \geq 4 \Rightarrow \frac{|z-i|^2}{|z+i|^2} \geq 4$$

$$\Rightarrow \frac{(z-i)\overline{(z-i)}}{(z+i)\overline{(z+i)}} \geq 4$$

$$\Rightarrow \frac{(z-i)(\bar z-\bar i)}{(z+i)(\bar z+\bar i)} \geq 4$$

$$\Rightarrow \frac{(z-i)(\bar z+i)}{(z+i)(\bar z-i)} \geq 4$$

$$\Rightarrow (z-i)(\bar z+i) \geq 4(z+i)(\bar z-i)$$

$$\Rightarrow z\bar z + iz - i\bar z + 1 \geq 4[z\bar z - iz + i\bar z + 1]$$

$$\Rightarrow z\bar z + i(z-\bar z) + 1 \geq 4[z\bar z + i(\bar z - z) + 1]$$

$$\Rightarrow 0 \geq 4z\bar z + 4i(\bar z - z) + 4 - z\bar z + i(\bar z - z) - 1$$

$$\Rightarrow 3z\bar z + 5i(\bar z - z) + 3 \leq 0$$

$$\Rightarrow 3(x^2+y^2) + 5i(-2iy) + 3 \leq 0$$

$$\Rightarrow 3x^2 + 3y^2 + 10y + 3 \leq 0$$

It represents the interior and frontier (or boundary) of the circle $3x^2+3y^2+10y+3=0$