3!

(1)

Ex. 3 In any spherical triangle ABC, Prove that

(i) $$\frac{\sin(A+B)}{\sin C} = \frac{\cos a + \cos b}{1+\cos c}$$

(ii) $$\frac{\sin(a+b)}{\sin c} = \frac{\cos A + \cos B}{1 - \cos C}$$

Sol (i) By Cosine formula, we have

$$\cos A = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

$$\Rightarrow \cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cos A \quad \text{---} \; (1)$$

$$\& \cos B = \frac{\cos b - \cos a \cdot \cos c}{\sin a \cdot \sin c}$$

$$\Rightarrow \cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B \quad \text{---} \; (2)$$

Adding (1) & (2), we get

$$\cos a + \cos b = \cos c(\cos a + \cos b) + \sin c \, (\sin b \cdot \cos A + \sin a \cdot \cos B)$$

$$\Rightarrow (\cos a + \cos b)(1 - \cos c) = \sin c \, (\sin b \cdot \cos A + \sin a \cdot \cos B) \quad \text{---} \; (3)$$

By Sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\therefore \sin b = \frac{\sin B}{\sin C} \cdot \sin c \quad \& \quad \sin a = \frac{\sin A}{\sin C} \cdot \sin c$$

Putting these values of $\sin a$ & $\sin b$ in (3), we get

$$(\cos a + \cos b)(1 - \cos c) = \sin c \cdot \left[\frac{\sin B}{\sin C} \cdot \sin c \cdot \cos A + \frac{\sin A}{\sin C} \cdot \sin c \cdot \cos B\right]$$

$$= \frac{\sin^2 c}{\sin C} \left[\sin A \cdot \cos B + \cos A \cdot \sin B\right]$$

$$= \frac{\sin^2 c}{\sin C} \cdot \sin(A+B)$$

(2)

$$\therefore \frac{\sin(A+B)}{\sin C} = \frac{(\cos a+\cos b)(1-\cos c)}{\sin^2 c}$$

$$= \frac{(\cos a+\cos b)(1-\cos c)}{1-\cos^2 c} = \frac{(\cos a+\cos b)\cancel{(1-\cos c)}}{\cancel{(1-\cos c)}(1+\cos c)}$$

$$= \frac{\cos a+\cos b}{1+\cos c}$$

Hence $\dfrac{\sin(A+B)}{\sin C} = \dfrac{\cos a+\cos b}{1+\cos c}$ Proved

ii) By Supplemental cosine formula, we have

$$\cos A = -\cos B.\cos C + \sin B.\sin C.\cos a$$

$$\cos B = -\cos A.\cos C + \sin A.\sin C.\cos b$$

$$\therefore \cos A+\cos B = -(\cos A+\cos B).\cos C + \sin C\left[\sin B.\cos a + \sin A.\cos b\right]$$

$$\Rightarrow (\cos A+\cos B)(1+\cos C) = \sin C.(\sin B.\cos a + \sin A.\cos b) \quad \text{— (1)}$$

Now, by sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\therefore \sin B = \frac{\sin b}{\sin c}.\sin C \quad \& \quad \sin A = \frac{\sin a}{\sin c}.\sin C$$

Using these in (1), we have

$$(\cos A+\cos B)(1+\cos C) = \sin C.\left[\frac{\sin b}{\sin c}.\sin C.\cos a + \frac{\sin a}{\sin c}.\sin C.\cos b\right]$$

$$= \frac{\sin^2 C}{\sin c}\left[\sin a.\cos b + \cos a.\sin b\right]$$

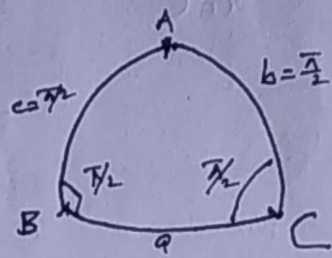
$$\Rightarrow \frac{(\cos A+\cos B)(1+\cos C)}{\sin^2 C} = \frac{1}{\sin c}.\sin(a+b) = \frac{\sin(a+b)}{\sin c}$$

$$\Rightarrow \frac{\cos A+\cos B}{1-\cos C} = \frac{\sin(a+b)}{\sin c}$$

Proved

(3)

<u>Example</u>. If $b+c=\pi$, Show That $\sin 2B + \sin 2C = 0$



By sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = K \text{ (say)}$$

$$\therefore \left.\begin{array}{l} \sin B = K \sin b \\ \sin C = K \sin c \end{array}\right] \longleftarrow (1)$$

Also, by Cosine formula, we have

$$\left.\begin{array}{l} \cos B = \dfrac{\cos b - \cos a . \cos c}{\sin a . \sin c} \\[2ex] \cos C = \dfrac{\cos c - \cos a . \cos b}{\sin a . \sin b}, \end{array}\right] - (2)$$

Now, $\sin 2B + \sin 2C$

$$= 2\sin B . \cos B + 2 \sin C . \cos C = 2\left[\sin B . \cos B + \sin C . \cos C\right]$$

$$= 2\left[K \sin b . \frac{\cos b - \cos a . \cos c}{\sin a . \sin c} + K \sin C . \frac{\cos c - \cos a . \cos b}{\sin a . \sin b}\right]$$

$$= 2K\left[\cancel{\sin c} . \frac{\cos b - \cos a . \cos c}{\sin a . \cancel{\sin c}} + \cancel{\sin b} \frac{\cos c - \cos a . \cos b}{\sin a . \cancel{\sin b}}\right]$$

$$\left[\begin{array}{l} \because b+c=\pi \\ b = \pi - c \\ \sin b = \sin(\pi - c) = \sin c \end{array}\right]$$

$$= \frac{2K}{\sin a}\left[\cos b - \cos a . \cos c + \cos c - \cos a . \cos b\right]$$

$$\begin{array}{l} \because b+c=\pi \\ b = \pi - c \\ \cos b = \cos(\pi - c) = -\cos c \end{array}$$

$$= \frac{2K}{\sin a}\left[-\cos c - \cos a . \cos c + \cos c - \cos a . (-\cos c)\right]$$

$$= \frac{2K}{\sin a}\left[-\cancel{\cos c} - \cancel{\cos a . \cos c} + \cancel{\cos c} + \cancel{\cos a . \cos c}\right]$$

$$= \frac{2K}{\sin a} \times 0 = 0$$ <u>Proved.</u>