Delambre's Analogies.

(i) $\dfrac{\sin\left(\frac{A+B}{2}\right)}{\cos\frac{C}{2}} = \dfrac{\cos\left(\frac{a-b}{2}\right)}{\cos c/2}$ (ii) $\dfrac{\sin\left(\frac{A-B}{2}\right)}{\cos C/2} = \dfrac{\sin\left(\frac{a-b}{2}\right)}{\sin c/2}$

(iii) $\dfrac{\cos\left(\frac{A+B}{2}\right)}{\sin\frac{C}{2}} = \dfrac{\cos\left(\frac{a+b}{2}\right)}{\cos\frac{c}{2}}$ (iv) $\dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}} = \dfrac{\sin\left(\frac{a+b}{2}\right)}{\sin\frac{c}{2}}$.

Proof. We know that

$$\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \cdot \sin c}} \text{ etc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s \cdot \sin(s-a)}{\sin b \cdot \sin c}} \text{ etc}$$

$$\therefore \sin\left(\frac{A+B}{2}\right) = \sin\frac{A}{2}\cdot\cos\frac{B}{2} + \cos\frac{A}{2}\cdot\sin\frac{B}{2}$$

$$= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \cdot \sin c}} \cdot \sqrt{\frac{\sin s \sin(s-b)}{\sin a \cdot \sin c}} + \sqrt{\frac{\sin s \cdot \sin(s-a)}{\sin b \cdot \sin c}} \cdot \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a \cdot \sin c}}$$

$$= \frac{\sin(s-b)}{\sin c}\cdot\sqrt{\frac{\sin s \cdot \sin(s-c)}{\sin a \cdot \sin b}} + \frac{\sin(s-a)}{\sin c}\cdot\sqrt{\frac{\sin s \cdot \sin(s-c)}{\sin a \cdot \sin b}}$$

$$= \left[\frac{\sin(s-b)+\sin(s-a)}{\sin c}\right]\cdot\sqrt{\frac{\sin s \cdot \sin(s-c)}{\sin a \cdot \sin b}}$$

$$= \frac{2\sin\frac{s-b+s-a}{2}\cdot\cos\frac{s-a-s+b}{2}}{2\sin c/2 \cdot \cos c/2}\cdot\cos\frac{C}{2}$$

$$= \frac{\sin\frac{a+b+c-a-b}{2}\cos\frac{(a-b)}{2}}{\sin\frac{c}{2}\cdot\cos\frac{c}{2}}\cdot\cos\frac{C}{2} = \frac{\cos\left(\frac{a-b}{2}\right)}{\cos\frac{c}{2}}\cdot\cos\frac{C}{2}$$

Hence the result.

In a similar fashion, we can prove other results.

Examples.

Ex. 1. In any spherical triangle ABC, show that

$$\tan\left(\frac{A-a}{2}\right).\tan\left(\frac{B+b}{2}\right) = \tan\left(\frac{B-b}{2}\right).\tan\left(\frac{A+a}{2}\right)$$

Soln. By Sine formula, we know that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b},$$

By Componendo & dividendo, we have

$$\frac{\sin A+\sin a}{\sin A-\sin a} = \frac{\sin B+\sin b}{\sin B-\sin b},$$

$$\Rightarrow \frac{2\sin\frac{A+a}{2}.\cos\frac{A-a}{2}}{2\cos\frac{A+a}{2}.\sin\frac{A-a}{2}} = \frac{2\sin\frac{B+b}{2}.\cos\frac{B-b}{2}}{2\cos\frac{B+b}{2}.\sin\frac{B-b}{2}}$$

$$\Rightarrow \frac{\tan\frac{A+a}{2}}{\tan\frac{A-a}{2}} = \frac{\tan\frac{B+b}{2}}{\tan\frac{B-b}{2}}$$

$$\Rightarrow \tan\left(\frac{A-a}{2}\right).\tan\left(\frac{B+b}{2}\right) = \tan\left(\frac{B-b}{2}\right)\tan\left(\frac{A+a}{2}\right)$$

Proved.

Ex. 2 In a spherical triangle ABC if $A=a$, show that

$$\tan\frac{a}{2} = \frac{\tan\left(\frac{b}{2}\right) \sim \tan\left(\frac{c}{2}\right)}{1-\tan\left(\frac{b}{2}\right).\tan\left(\frac{c}{2}\right)}$$

By Cosine formula, we have

$$\cos A = \frac{\cos a - \cos b.\cos c}{\sin b.\sin c}$$

Given that $A=a$

$$\therefore \cos a = \frac{\cos a - \cos b.\cos c}{\sin b.\sin c}$$

$$\Rightarrow \cos a - \cos b.\cos c = \cos a.\sin b.\sin c$$

$$\Rightarrow \cos a\,(1-\sin b.\sin c) = \cos b.\cos c$$

$$\therefore \cos a = \frac{\cos b \cdot \cos c}{1 - \sin b \cdot \sin c}$$

$$\Rightarrow \frac{\cos a}{1} = \frac{\cos b \cdot \cos c}{1 - \sin b \cdot \sin c}$$

By Componendo & dividendo, we have

$$\frac{1 - \cos a}{1 + \cos a} = \frac{1 - \sin b \cdot \sin c - \cos b \cdot \cos c}{1 - \sin b \cdot \sin c + \cos b \cdot \cos c}$$

$$\Rightarrow \frac{\cancel{2} \sin^2 \frac{a}{2}}{\cancel{2} \cos^2 \frac{a}{2}} = \frac{1 - (\cos b \cdot \cos c + \sin b \cdot \sin c)}{1 + (\cos b \cdot \cos c - \sin b \cdot \sin c}$$

$$= \frac{1 - \cos(b \sim c)}{1 + \cos(b + c)} = \frac{\cancel{2} \sin^2 \left(\frac{b \sim c}{2}\right)}{\cancel{2} \cos^2 \left(\frac{b + c}{2}\right)}$$

$$\Rightarrow \tan^2 \frac{a}{2} = \frac{\sin^2 \left(\frac{b \sim c}{2}\right)}{\cos^2 \left(\frac{b + c}{2}\right)}$$

$$\therefore \tan \frac{a}{2} = \frac{\sin \left(\frac{b \sim c}{2}\right)}{\cos \left(\frac{b + c}{2}\right)}$$

$$= \frac{\sin \frac{b}{2} \cdot \cos \frac{c}{2} \sim \cos \frac{b}{2} \cdot \sin \frac{c}{2}}{\cos \frac{b}{2} \cdot \cos \frac{c}{2} - \sin \frac{b}{2} \cdot \sin \frac{c}{2}}$$

$$= \frac{\dfrac{\sin \frac{b}{2} \cdot \cancel{\cos \frac{c}{2}}}{\cos \frac{b}{2} \cdot \cancel{\cos \frac{c}{2}}} \sim \dfrac{\cancel{\cos \frac{b}{2}} \cdot \sin \frac{c}{2}}{\cancel{\cos \frac{b}{2}} \cdot \cos \frac{c}{2}}}{\dfrac{\cancel{\cos \frac{b}{2} \cdot \cos \frac{c}{2}}}{\cancel{\cos \frac{b}{2} \cdot \cos \frac{c}{2}}} - \dfrac{\sin \frac{b}{2}}{\cos \frac{b}{2}} \cdot \dfrac{\sin c/2}{\cos c/2}}$$

$$\tan \frac{a}{2} = \frac{\tan\left(\frac{b}{2}\right) \sim \tan\left(\frac{c}{2}\right)}{1 - \tan\left(\frac{b}{2}\right) \cdot \tan\left(\frac{c}{2}\right)}$$

Proved.