Subject	Class	Paper	Study materials on	Resource Person
Mathematics	D-I(H)	I	Transformation of Eqn	Dr. S. Ahmad

Example If α, β, γ be the roots of the eqn $x^3+px^2+qx+r=0$, find the equation whose roots are

(i) $\frac{\alpha}{\beta+\gamma}, \frac{\beta}{\gamma+\alpha}, \frac{\gamma}{\alpha+\beta}$

(ii) $\alpha(\beta+\gamma), \beta(\gamma+\alpha), \gamma(\alpha+\beta)$

Soln (i) Let $y = \frac{\alpha}{\beta+\gamma} = \frac{\alpha}{\alpha+\beta+\gamma-\alpha} = \frac{\alpha}{-p-\alpha}$

$\Rightarrow -y(p+\alpha) = \alpha \Rightarrow -yp - y\alpha = \alpha$

$\Rightarrow -yp = \alpha(1+y) \Rightarrow \alpha = -\frac{py}{1+y}$

Since α is a root of the given equation

$\therefore \alpha^3 + p\alpha^2 + q\alpha + r = 0$

$\Rightarrow \left(-\frac{py}{1+y}\right)^3 + p\cdot\left(\frac{-py}{1+y}\right)^2 + q\cdot\left(\frac{-py}{1+y}\right) + r = 0$

$\Rightarrow -\frac{p^3y^3}{(1+y)^3} + p\cdot\frac{p^2y^2}{(1+y)^2} - \frac{pqy}{1+y} + r = 0$

$\Rightarrow -p^3y^3 + p^3y^2(1+y) - pqy(1+y)^2 + r(1+y)^3 = 0$

$\Rightarrow -p^3y^3 + p^3y^2 + p^3y^3 - pqy(1+2y+y^2) + r(1+3y+3y^2+y^3)$

$\Rightarrow y^3[\cancel{-p^3} + \cancel{p^3} - pq + r] + y^2[p^3 - 2pq + 3r]$
$+ y[-pq + 3r] + r = 0$

$\Rightarrow \boxed{(r-pq)y^3 + (3r-2pq+p^3)y^2 + (3r-pq)y + r = 0}$

It is required equation.

(ii) Let $y = \alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - \beta\gamma = q - \beta\gamma = q - \frac{\alpha\beta\gamma}{\alpha}$

$= q - \frac{(-r)}{\alpha} = q + \frac{r}{\alpha}$

$\Rightarrow y - q = \frac{r}{\alpha}$

$\therefore \alpha = \frac{r}{y-q}$

Since α is a root of the equation $x^3+px^2+qx+r=0$

$\therefore \alpha^3 + p\alpha^2 + q\alpha + r = 0$

$\Rightarrow \left(\frac{r}{y-q}\right)^3 + p\cdot\left(\frac{r}{y-q}\right)^2 + q\cdot\left(\frac{r}{y-q}\right) + r = 0$

$\Rightarrow r^3 + pr^2(y-q) + rq(y-q)^2 + r(y-q)^3 = 0$

$\Rightarrow r^3 + pr^2(y-q) + rq(y^2-2yq+q^2) + r(y^3-3y^2q+3yq^2-q^3)$

$\Rightarrow r\left[r^2 + pr(y-q) + q(y^2-2yq+q^2) + (y^3-3y^2q+3yq^2-q^3)\right] = 0$

But $r \neq 0$

$\therefore r^2 + pr(y-q) + q(y^2-2yq+q^2) + (y^3-3y^2q+3yq^2-q^3) = 0$

$\Rightarrow y^3 + y^2(q-3q) + y(pr-2q^2+3q^2) + \cancel{q^3} - \cancel{q^3} + r^2 - prq = 0$

$$\Rightarrow \boxed{y^3 - 2qy^2 + (pr+q^2)y + (r^2-pq) = 0}$$

It is required equation.

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Mathematics	D-I(H)	I	Transformation of Eq^n	Dr. S. Ahmad

<u>Ex.</u> If α, β, γ be the roots of the cubic equation $x^3+px^2+qx+r=0$. Find the equation whose roots are $\alpha-\frac{1}{\beta\gamma}$, $\beta-\frac{1}{\gamma\alpha}$, $\gamma-\frac{1}{\alpha\beta}$.

<u>Sol^n.</u> Let $y = \alpha - \frac{1}{\beta\gamma} = \frac{\alpha\beta\gamma-1}{\beta\gamma} = \frac{-r-1}{\beta\gamma} = \frac{-(r+1)}{\beta\gamma} = \frac{-(r+1)\alpha}{\alpha\beta\gamma}$

$$= \frac{-(r+1)\alpha}{-r}$$

$$\Rightarrow y = \frac{(r+1)\alpha}{r}$$

$$\therefore \alpha = \frac{ry}{r+1}$$

Since α is a root of the equation $x^3+px^2+qx+r=0$

$$\therefore \alpha^3+p\alpha^2+q\alpha+r=0$$

$$\Rightarrow \left(\frac{ry}{r+1}\right)^3 + p\cdot\left(\frac{ry}{r+1}\right)^2 + q\left(\frac{ry}{r+1}\right) + r = 0$$

$$\Rightarrow r^3y^3 + pr^2y^2(r+1) + rqy(r+1)^2 + r(r+1)^3 = 0$$

$$\Rightarrow r\left[r^2y^3 + pry^2(r+1) + qy(r+1)^2 + (r+1)^3\right] = 0$$

But $r \neq 0$.

$$\Rightarrow \boxed{r^2y^3 + pr(r+1) + qy(r+1)^2 + (r+1)^3 = 0}$$

It is required equation.

<u>Ex.</u> If α, β, γ are the roots of $x^3-x-1=0$, find the equation whose roots are $\frac{1+\alpha}{1-\alpha}$, $\frac{1+\beta}{1-\beta}$, $\frac{1+\gamma}{1-\gamma}$. Hence write down the value of $\sum\left(\frac{1+\alpha}{1-\alpha}\right)$.

Solution

Let $y = \frac{1+\alpha}{1-\alpha}$

$\Rightarrow y(1-\alpha) = 1+\alpha$

$\Rightarrow y - y\alpha = 1+\alpha$

$\Rightarrow y-1 = y\alpha + \alpha = \alpha(y+1)$

$\therefore \alpha = \frac{y-1}{y+1}$

Since α is a root of the equation $x^3-x-1=0$

$\therefore \alpha^3 - \alpha - 1 = 0$

$\Rightarrow \left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right) - 1 = 0$

$\Rightarrow (y-1)^3 - (y-1)(y+1)^2 - (y+1)^3 = 0$

$\Rightarrow (y^3-3y^2+3y-1) - (y^2-1)(y+1) - (y^3+3y+3y^2+1) = 0$

$\Rightarrow (y^3-3y^2+3y-1) - (y^3+y^2-y-1) - (y^3+3y^2+3y+1) = 0$

$\Rightarrow (y^3-y^3-y^3) + (-3-1-3)y^2 + (3+1-3)y - 1 + 1 \cancel{+1} = 0$

$\Rightarrow -y^3 - 7y^2 + y \cancel{+1} = 0$

$\Rightarrow \boxed{y^3+7y^2-y+1=0}$

It is required equation

$\therefore \sum\left(\frac{1-\alpha}{1+\alpha}\right) = \frac{-7}{1} = -7$ Ans.