Subject	Class	Paper	Reading materials on	Resource Person
Mathematics	D-1 (H)	I	General properties of Equation	Dr. S. Ahmad Asso. Prof.

Theorem: Every equation of nth degree has n roots & no more.

Proof. Let $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$ — (1),
be an equation of nth degree.

By fundamental theorem of algebra, Eqn (1) has at least one root. Let α_1 be a such roots of equation $f(x) = 0$ and as such $f(x)$ must be divisible by $x - \alpha_1$.

Then by factor theorem, we have

$$f(x) = (x-\alpha_1)\,\varphi_1(x)$$

Where $\varphi_1(x)$ is a polynomial of $(n-1)$th degree.

∴ Again $\varphi_1(x) = 0$ must have a root say α_2 and as such $\varphi_1(x)$ must be divisible by $x-\alpha_2$

$$\therefore \varphi_1(x) = (x-\alpha_2)\,\varphi_2(x)$$

$$\Rightarrow f(x) = (x-\alpha_1)(x-\alpha_2)\,\varphi_2(x)$$

Where $\varphi_2(x)$ is a polynomial of $(n-2)$th degree.

Continuing this process as above, we have

$$f(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots\cdots(x-\alpha_n)\,\varphi_n(x)$$

Where $\varphi_n(x)$ is a polynomial of $(n-n)$ i.e. zero degree i.e. Constant.

Comparing the Coefficients of x^n on both sides, we have

$$a_0 = \varphi_n(x)$$

Hence $f(x) = a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ —— (2)

Now, if we put equal to any one of the n quantities $\alpha_1, \alpha_2, \dots \alpha_n$, then $f(x)$ vanishes and we can say $f(x) = 0$ has n roots. Again if x be given any value other than above n quantities, then $f(x)$ does not vanish and such any other quantity Cannot be a root of the equation. Thus we Come to the Conclusion That an equation of nth degree has n roots & no more.

Problem 1. Prove that $(1-a)(1-b)(1-c)\ldots\ldots = n$, if $1, a, b, c\ldots$ be the roots of the equation $x^n - 1 = 0$

Sol^n. Let $f(x) = x^n - 1 = 0$

Since $1, a, b, c, \ldots\ldots$ are the roots of the equation, then $f(x)$ must be divisible by $(x-1)(x-a)(x-b)(x-c)\ldots$

Hence, we must have

$$x^n - 1 \equiv (x-1)(x-a)(x-b)(x-c)\ldots$$

Diff. w.r. to x, we have

$$nx^{n-1} = (1-0)(x-a)(x-b)(x-c)\ldots + (x-1)\frac{d}{dx}\{(x-a)(x-b)(x-c)\ldots\}$$

Putting $x=1$, we have

$$n(1)^{n-1} = (1-a)(1-b)(1-c)\ldots + 0\cdot\frac{d}{dx}\{(x-a)(x-b)(x-c)\ldots\}$$

$$n = (1-a)(1-b)(1-c)\ldots$$

Hence $(1-a)(1-b)(1-c)\ldots = n$ Proved.

Problem 2. If $\alpha_1, \alpha_2, \ldots \alpha_n$ are the roots of $x^n + nax - b = 0$ Show that $(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n) = n(\alpha_1^{n-1}+a)$

Sol^n. We have

$$x^n + nax - b = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Diff. w.r. to x, we have

$$nx^{n-1} + na - 0 = (1-0)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) + (x-\alpha_1)\frac{d}{dx}\{(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)\}$$

Putting $x=\alpha_1$, we have

$$n\cdot\alpha_1^{n-1} + na = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n) + 0$$

$$\Rightarrow n(\alpha_1^{n-1}+a) = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)$$

Proved