Study materials of Mathematics for class D–III (H), Paper–VI on "Cauchy Schwarz's Inequality" Composed by Dr. S. Ahmed, Associate Professor, U. R. College Rosera,

## Normed linear Space

Defn: Let $V(F)$ be a linear space and let $\| \, \|$ be a function from $V$ into $R$ such that for all $u, v \in V$ and $\alpha \in F$, (Whether real or Complex), we have

$[n_1]$: $\|u\| > 0$

$[n_2]$: $\|u\| = 0 \Leftrightarrow u = 0$

$[n_3]$: $\|\alpha u\| = |\alpha| \, \|u\|$

$[n_4]$: $\|u+v\| \leq \|u\| + \|v\|$ (triangular inequality)

Then the function $\| \, \|$ is called a norm on $V$ and the pair $(V, \| \, \|)$ is called a normed linear space.

The real number $\|u\|$ is called the norm of the vector $u$.

By the first axiom $[IP_1]$ of an inner product, $(u, u)$ is non-negative for any vector $u$. Thus its positive square root exists. We use the notation $\|u\| = \sqrt{(u,u)}$

## Cauchy Schwarz's Inequality

Theorem: If $u$ and $v$ be two vectors in an inner product space, then $|(u,v)| \leq \|u\| \, \|v\|$

**Proof**: If $v = 0$, then $\|v\| = 0$ and $|(u,v)| = 0$. Therefore in this case, both sides vanish and the result is clear.

Now, let $v \neq 0$. Then for any scalar $\lambda$, we have

$$(u+\lambda v, u+\lambda v) \geq 0 \; ; \text{ since } (w,w) = \|w\|^2 \geq 0 \quad \forall\, w \in V$$

$$\Rightarrow (u, u+\lambda v) + \lambda (v, u+\lambda v) > 0$$

by linearity of inner product.

$$\Rightarrow (u,u) + \bar{\lambda}(u,v) + \lambda\{(v,u) + \bar{\lambda}(v,v)\} \geq 0$$

since $(u, bv+cw)$
$= \bar{b}(u,v) + \bar{c}(u,w)$

$$\Rightarrow \|u\|^2 + \bar{\lambda}(u,v) + \lambda(v,u) + \lambda\bar{\lambda}(v,v) \geq 0 \quad \text{— (1)}$$

Since $v \neq 0 \Rightarrow \|v\| \neq 0$

therefore putting $\lambda = -\dfrac{(u,v)}{\|v\|^2}$ in (1), we get

$$\|u\|^2 - \frac{\overline{(u,v)}}{\|v\|^2}(u,v) - \frac{(u,v)}{\|v\|^2}(v,u) + \frac{|(u,v)|^2}{(\|v\|^2)^2}\cdot\|v\|^2 \geq 0.$$

$$\Rightarrow \|u\|^2 - \frac{|(u,v)|^2}{\|v\|^2} - \frac{(u,v)\,\overline{(u,v)}}{\|v\|^2} + \frac{|(u,v)|^2}{\|v\|^2} \geq 0$$

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$$\Rightarrow \|u\|^2\|v\|^2 - |(u,v)|^2 \geq 0$$

$$\Rightarrow |(u,v)| \leq \|u\|\cdot\|v\|$$ Hence the result.

Theorem: Every inner product vector space is a normed vector space.

Proof: Let $V$ be an inner product space. In order to prove that it is a normed vector space, we have to show that it satisfies the properties of normed vector space.

If $u \in V$, then by defn $\|u\|^2 = (u, u)$

(i) $\|u\|^2 = (u,u) \geq 0$ by the property of inner product space.

Also, $(u,u) = 0$ iffs $u = 0$

$\therefore \|u\|^2 = 0$ and $\|u\| = 0$ iff $u = 0$

(ii) $\|\alpha u\|^2 = (\alpha u, \alpha u)$

$= \alpha\bar{\alpha}\,(u,u)$

$\therefore \|\alpha u\|^2 = |\alpha|^2 \|u\|^2$

Taking square root of both sides, we get

$\|\alpha u\| = |\alpha|\,\|u\|$

(iii) $\|u+v\|^2 = (u+v, u+v)$.

$= (u, u+v) + (v, u+v)$

$= (u,u) + (u,v) + (v,u) + (v,v)$

$= \|u\|^2 + (u,v) + (v,u) + \|v\|^2$

$$= \|u\|^2 + \|v\|^2 + (u,v) + (v,u)$$

$$= \|u\|^2 + \|v\|^2 + (u,v) + \overline{(u,v)}$$

$$= \|u\|^2 + \|v\|^2 + 2 \text{ real part of } (u,v) \quad \text{——— (1)}$$

Now, if $w = a+ib$ be a complex number, then

$$|w| = \sqrt{a^2+b^2} \text{ and the real part of } w = a$$

$\therefore$ real part of $w \leq |w|$

Hence from (1), we have

$$\|u+v\|^2 \leq \|u\|^2 + \|v\|^2 + 2\,|(u,v)|$$

By Cauchy Schwarz's inequality, we have

$$|(u,v)| \leq \|u\| \cdot \|v\|$$

Hence $\|u+v\|^2 \leq \|u\|^2 + \|v\|^2 + 2\|u\|\cdot\|v\|$

$$= [\|u\| + \|v\|]^2$$

$$\Rightarrow \|u+v\| \leq \|u\| + \|v\|$$

Thus all the properties of a normed vector space are satisfied.

Hence every inner product space is a normed vector space.